RESEARCH ARTICLE

A fractional differential approach to plant pest mature and immature biological enemy

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The issue of plant pests is a critical area of investigation at present. While numerous chemical solutions are available, their toxic effects make the use of natural enemies a preferable alternative. This article introduces a fractional calculus approach to model the dynamics between mature and immature natural enemies of plant pests. The study discusses the existence and uniqueness of the solution, the non-negativity of the solution and both global and local stability of the equilibrium point. Given that memory is intrinsic to biological systems, the proposed technique is advantageous due to the memory effect of fractional derivatives, leading to more accurate solutions. Simulations with different fractional parameters generate optimal solutions.

Key Words: *Fractional differential; Plant pest natural enemy; Numerical simulations*

INTRODUCTION

In the present era, the plant-pest problem is a significant global issue. Discussions about plant pests date back to the early 18th century, with George and Watt addressing the pests and delights of tea plants in 1898 and Hilderic Friend studying worms as plant pests in 1911 [1,2]. Over time, many authors have proposed solutions to plant-pest problems, including chemical solutions, biological enemies and pest infections.

Numerous chemical solutions have been explored. For instance, Choo Ho, et al. discussed the effects of some chemical pesticides in 1998 and Hajji et al. examined nanobased pesticides in 2021 [3,4]. However, chemical solutions can sometimes be harmful to both plants and humans. BN Aloo et al. highlighted the adverse effects of agrochemicals on beneficial plant rhizobacteria in agricultural systems in 2021 [5].

Biological enemies are a preferable solution to plant-pest problems. The study of biological control began in 1888 when the Vedalia bug was introduced in California from Australia [6]. Paul DeBach and David Rosen's 1991 book on biological control explains how natural enemies can biologically control pest populations [7]. In 1995, Thomas et al. published research on the biological control of grasshoppers by fungi [8].

Mathematical modeling of plant-pest interactions has gained traction in recent years. Haith et al. studied models for pest management in the analysis of potato integration in 1987 [9]. Maiti et al. developed a mathematical model on the usefulness of biocontrol for pests in tea in 2008 [10]. Kumar et al. investigated a mathematical model on plant pests and natural enemies with twin gestation delays as a biological control technique in 2018. Many researchers have attempted to construct robust models by addressing various real-world challenges to mitigate flaws in predator-prey relationship models.

Currently, many plant-pest models are developed using ordinary differential equations with various parameters, including delay, harvesting, etc. The fractional differentiation approach for plant-pest models is a new research area. Fractional differentiation introduces a memory effect, which is a valuable criterion for solving plant-pest problems. Bhattacharya et al. studied a fractional differential approach with memory for the paradox of enrichment in 2013. Samanta et al. developed a fractional-order prey-predator model incorporating prey in 2018 and Moustafa et al. wrote a paper on a fractional-order prey-predator model in 2019.

Several definitions of fractional differential equations exist, including Grunwald-Letnikov, Riemann-Liouville and Caputo. Among these, Caputo's fractional derivative is the most popular and well-developed, with initial conditions similar to those for integerorder derivatives.

Recently, researchers have increasingly focused on fractional differentiation due to its memory terms and properties. Caputo fractional derivatives have gained interest for their applications in modeling pandemics like COVID-19, electrical engineering, biochemistry (e.g., modeling proteins and polymers), acoustics, material modeling, rheology and mechanical systems. The fundamental properties and applications of fractional derivatives can be found in the works. Given that memory is inherent in biological systems, fractional derivatives are particularly relevant. This is why I am studying a fractional-order plant-pest model involving immature and mature biological enemies. For fractional differential based model one can cite the following papers.

In this research paper, we discussed a fractional-order differential model for plant pest mature and immature biological enemies.

MATERIALS AND METHODS

Using a fractional derivative, we examined the food chain dynamics of plant pests with their immature and mature biological enemies in this work. In this case, we presented the following mathematical approach:

- Plant ($x(t)$), Pest ($y(t)$), premature biological enemy ($z_1(t)$) and mature biological enemy ($z_2(t)$) are the four categories of species.
- With an inherent progress rate of γ_0 and a carrying capacity of γ_0/β_0, the plants grow logistically. The per capita progress rate of the plants is $x(\gamma_0-\beta_0 x)$ when there is no pest population.
- Pests harvest plants and the holing type-1 functional response is the result of this harvesting.
- The pest is cropped by a mature biological enemy with a holing type-1 functional reaction.
- Let γ_1 represent the cropping rates of plants by pests, γ_2 represent the progress rate of pests by plants, α_1 represent the cropping rate of pests by mature biological enemies and α_2 represent the progress rate of immature biological enemies by mature biological enemies and pests. The mortality rates of pest, immature biological enemy, and mature biological enemy are d, d_1 and d_2, respectively. The pace at which immature biological enemies evolve into mature biological enemies is α_3

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and the rate at which mature biological enemies proceed through immature biological enemies is α_3.

Plant pest premature-mature biological enemy

$$\begin{aligned} {}^cD^\delta x &= x(\gamma_0 - \beta_0 x - \gamma_1 y), \\ {}^cD^\delta y &= y(\gamma_2 x - d - \alpha_1 z_2), \\ {}^cD^\delta z_1 &= \alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1, \\ {}^cD^\delta z_2 &= \alpha_3 z_1 - d_2 z_2, \end{aligned} \tag{1.1}$$

With initial conditions: $x(0)>0$, $y(0)>0$, $z_1(0)>0$ and $z_2(0)>0$.

The following is a summary of our work: We reviewed some basic fractional derivative definitions in the preliminaries section. The existence and uniqueness results for the system (1.1) are derived in the main results section. For the system (1.1), the equilibrium points and their stability analysis are also performed. Numerical analysis is performed at the end of this work using matlab code fde12 for fractional differential equations.

Preliminaries

We will review certain terminology and basic fractional calculus results in this section, this will be used for the duration of the research.

Definition: The fractional integral of a function ϕ with order $\delta>0$ lower bound zero is defined as follows.

$$I^\delta \phi(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-\eta)^{\delta-1} \phi(\eta) d\eta, \quad t>0,$$

and $I^0\phi(t) := \phi(t)$, where the Euler Gamma function is $\Gamma(\cdot)$. For $b>0$, this fractional integral satisfies the conditions $I^\delta \circ I^b = I^{\delta+b}$.

Definition: The Riemann-Liouville fractional derivative of a function ϕ with the lower limit zero of order $\delta>0$ is given by

$$D^\delta \phi(t) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{dt^n} \int_0^t (t-\eta)^{n-\delta-1} \phi(\eta) d\eta,$$

where $n-1<\delta<n, n \in \mathbb{N}$, up to order $(n-1)$, the function $\phi(t)$ has an absolutely continuous derivative. Moreover $D^0\phi(t) = \phi(t)$ and $D^\delta I^\delta \phi(t) = \phi(t)$ for $t>0$.

Definition: The caputo fractional derivative of a function $\phi \in C^n([0,\infty))$ with the lower limit zero of order $\delta>0$ is given by

$${}^cD^\delta \phi(t) = \frac{1}{\Gamma(n-\delta)} \int_0^t (t-\eta)^{n-\delta-1} \frac{d^n}{d\eta^n} \phi(\eta) d\eta,$$

where $n-1<\delta<n, n \in \mathbb{N}$.

Consider the system

$${}^CD^\delta f(t) = g(f,t)$$

with the initial condition: $f(t_0)=ft_0$, where $\delta \in (0,1]$, g: $\Gamma \times [t_0, \infty) \to R^n, \Gamma \subset R^n$, if $g(f,t)$ satisfies the local Lipschitz condition with respect to f, then there exist a unique solution of (2.1) on $\Gamma \times [t_0, \infty)$.

RESULTS AND DISCUSSION

Existence and uniqueness

Theorem 3.1. *consider the region $\Gamma \times [t_0, T)$ where $\Gamma = \{(x,y,z_1,z_2) \in \mathbb{R}^n : max\{|x|,|y|,|z_1|,|z_2|\} < M\}$. where $T<\infty$, $X=(x,y,z_1,z_2)$and $\bar{X} = (\bar{x},\bar{y},\bar{z}_1,\bar{z}_2)$ Then for each $X(t_0) = (x_{t_0}, y_{t_0}, z_{1_{t_0}}, z_{2_{t_0}}) \in \Gamma$, there exist a unique solution $X(t) \in \Gamma$ of system (1.1) with initial condition $X(t_0)$, which is defined for all $t \geq t_0$.*

Proof. Now consider a relation $\rho(X) = (\rho_1(X), \rho_2(X), \rho_3(X), \rho_4(X))$, Where

$$\begin{aligned} \rho_1(X) &= x(\gamma_0 - \beta_0 x - \gamma_1 y) \\ \rho_2(X) &= y(\gamma_2 x - d - \alpha_1 z_2 - \alpha_2 z_1) \\ \rho_3(X) &= \alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1 \\ \rho_4(X) &= \alpha_3 z_1 - d_2 z_2. \end{aligned}$$

$\forall X, \bar{X} \in \Gamma$, now,

$$\|\rho(X) - \rho(\bar{X})\| = |\rho_1(X) - \rho_1(\bar{X})| + |\rho_2(X) - \rho_2(\bar{X})| + |\rho_3(X) - \rho_3(\bar{X})| + |\rho_4(X) - \rho_4(\bar{X})|.$$

$$\begin{aligned} &= |x(\gamma_0 - \beta_0 x - \gamma_1 y) - \bar{x}(\gamma_0 - \beta_0 \bar{x} - \gamma_1 \bar{y})| + |y(\gamma_2 x - d - \alpha_1 z_2) - \bar{y}(\gamma_2 \bar{x} - d - \alpha_1 \bar{z}_2)| \\ &\quad + |\alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1 - (\alpha_2 \bar{y} \bar{z}_2 - \alpha_3 \bar{z}_1 - d_1 \bar{z}_1)| + |\alpha_3 z_1 - d_2 z_2 - \alpha_3 \bar{z}_1 - d_2 \bar{z}_2|. \end{aligned}$$

$$\begin{aligned} &= |\gamma_0(x-\bar{x}) - \beta_0(x^2 - \bar{x}^2) - \gamma_1(xy - \bar{x}\bar{y})| + |\gamma_2(xy - \bar{x}\bar{y}) - d(y-\bar{y}) - \alpha_1(yz_2 - \bar{y}\bar{z}_2)| \\ &\quad + |\alpha_2(z_2 y - \bar{z}_2 \bar{y}) - d_1(z_1 - \bar{z}_1) - \alpha_3(z_1 - \bar{z}_1)| + |\alpha_3(z_1 - \bar{z}_1) - d_2(z_2 - \bar{z}_2)|. \end{aligned}$$

$$\begin{aligned} \|\rho(X) - \rho(\bar{X})\| &\leq \gamma_0|x-\bar{x}| + \beta_0 2M|x-\bar{x}| + \gamma_1 M|x-\bar{x}| + \gamma_2|y-\bar{y}|M + d|y-\bar{y}| + \alpha_1 M|y-\bar{y}| \\ &\quad + \alpha_2 M|z_2 - \bar{z}_2| + d_1|z_1 - \bar{z}_1| + \alpha_3|z_1 - \bar{z}_1| + \alpha_3|z_1 - \bar{z}_1| + d_2|z_2 - \bar{z}_2|. \end{aligned}$$

$$= (\gamma_0 + 2M\beta_0 + M\gamma_1)|x-\bar{x}| + (M\gamma_2 + d + M\alpha_1)|y-\bar{y}| + (d_1 + 2\alpha_3)|z_1 - \bar{z}_1| + (M\alpha_2 + d_2)|z_2 - \bar{z}_2|.$$

Assume-

$$E = max\{\gamma_0 + 2M\beta_0 + M\gamma_1, M\gamma_2 + d + M\alpha_1, d_1 + 2\alpha_3, M\alpha_2 + d_2\}$$
$$\|\rho(X) - \rho(\bar{X})\| \leq E\|X - \bar{X}\|$$

As a result, $\rho(X)$ satisfies the Lipschitz condition with regard to X and hence there exists a unique solution X(t) for system.

Non-negativity and uniform boundedness

The solutions of the system which (1.1) starts in R^4_+ and non-negative and uniform bounded. Proof. Applied the results used in. Suppose

$$\chi(t) = x(t) + \frac{\gamma_1}{\gamma_2} y(t) + \frac{\alpha_1\gamma_1}{\alpha_2\gamma_2} z_1 + \frac{\alpha_1\gamma_1}{\alpha_2\gamma_2} z_2, \tag{3.1}$$

then

$$\begin{aligned} {}^cD^\delta \chi(t) &= {}^cD^\delta x(t) + (\frac{\gamma_1}{\gamma_2}) {}^cD^\delta y(t) + (\frac{\alpha_1\gamma_1}{\alpha_2\gamma_2}) {}^cD^\delta z_1 + (\frac{d_1\alpha_1\gamma_1}{\alpha_2\gamma_2}) {}^cD^\delta z_2, \\ &= x(\gamma_0 - \beta_0 x) - \frac{d\gamma_1}{\gamma_2} y - \frac{d_1\alpha_1\gamma_1}{\alpha_2\gamma_2} z_1 - \frac{d_2\alpha_1\gamma_1}{\alpha_2\gamma_2} z_2, \\ {}^cD^\delta \chi(t) + p\chi(t) &= (p+\gamma_0)x - \beta_0 x^2 + \frac{\gamma_1}{\gamma_2}(p-d)y(t) + \frac{\alpha_1\gamma_1}{\alpha_2\gamma_2}(p-d_1)z_1 + \frac{d_1\alpha_1\gamma_1}{\alpha_2\gamma_2}(p-d_2)z_2, \end{aligned}$$

Let us assume $p < min\{d, d_1, d_2\}$, then

$$\begin{aligned} {}^cD^\delta \chi(t) + p\chi(t) &\leq -\beta_0(x - \frac{p+\gamma_0}{2\beta_0})^2 + \frac{(p+\gamma_0)^2}{4\beta_0}, \\ &\leq \frac{(p+\gamma_0)^2}{4\beta_0}, \end{aligned}$$

Now applying the comparison theorem for fractional order, one reach

$$0 \leq \chi(t) \leq \chi(0) E_\delta(-pt^\delta) + \frac{(p+\gamma_0)^2}{4\beta_0}(t^\delta) E_{\delta,\delta+1}(-p(t)^\delta).$$

Where E_δ and $E_{\delta,\ \delta+1}$ are mittag leffler function. Now by taking $t \to \infty$ one get

$$0 \leq \chi(t) \leq \frac{(p+\gamma_0)^2}{4p\beta_0},$$

Hence, the solutions of system of fractional differential equation begins in R^4_+ are uniformly bounded with in the region $\chi 1$ defined as

$$\chi_1 = \{(x, y, z_1, z_2) \in \mathbb{R}^4_+ : \chi(t) \leq \frac{(p+\gamma_0)^2}{4p\beta_0} + \omega, \omega > 0\}. \tag{3.2}$$

Now we will prove that the solution of fraction order system is non-negative. Consider the first equation of system of equation.

$$^cD^\delta x = x(\gamma_0 - \beta_0 x - \gamma_1 y),$$

From equation 3.1 and 3.2 and considering $\omega \to 0$ one get

$$\chi(t) = x(t) + \frac{\gamma_1}{\gamma_2}y(t) + \frac{\alpha_1\gamma_1}{\alpha_2\gamma_2}z_1 + \frac{d_1\alpha_1\gamma_1}{\alpha_2\gamma_2}z_2 \leq \frac{(p+\gamma_0)^2}{4p\beta_0} = \tau, \tag{3.3}$$

From **??** and 3.3 one get

$$\begin{aligned} ^cD^\delta x &\geq x(\gamma_0 - \beta_0\tau - \gamma_1\tau), \\ ^cD^\delta x &\geq \tau_1 x, \end{aligned}$$

where $\tau_1=(\gamma_0-\beta_0\tau-\gamma_1\tau)$, from comparison theorem of one get

$$x \geq x_0 E_{\delta,1}(\tau_1 t^\delta).$$

Hence $x \geq 0$. Now on taking equation 2 of system 1.1 and 3.1 one get

$$\begin{aligned} ^cD^\delta y &\geq y(\gamma_2 x - d - \alpha_1\tau), \\ ^cD^\delta y &\geq y(-d - \alpha_1\tau)\{\because x \geq 0\}, \\ &\geq -\tau_2 y, \end{aligned}$$

where $\tau_2=d+\alpha_1\tau$, Hence

$$\begin{aligned} y &\geq y_0 E_{\delta,1}(-\tau_2 t^\delta), \\ &\implies y \geq 0. \end{aligned}$$

Now considering the equation 3 of system of equation 1.1 one get

$$\begin{aligned} cD^\delta z_1 &= \alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1, \\ &\geq -(\alpha_3 + d_1) z_1 \{\because y \geq 0\}, \\ &\geq -\tau_3 z_1, where, \tau_3 = \alpha_3 + d_1, \\ &\implies z_1 \geq z_{10} E_{\delta,1}(-\tau_3 t^\delta), \\ &\implies z_1 \geq 0. \end{aligned}$$

Again from equation 4 of system of equation 1.1 one get

$$\begin{aligned} ^cD^\delta z_2 &= \alpha_3 z_1 - d_2 z_2, \\ &\geq -d_2 z_2 \{\because z_1 \geq 0\}, \\ &\implies z_2 \geq z_{20} E_{\delta,1}(-d_2 t^\delta), \\ &\implies z_2 \geq 0. \end{aligned}$$

Hence the system of equation 1.1 have non-negative solutions.

Basic reproduction number

For finding equilibrium points and their stability we will introduce basic reproduction with the help of pest free equilibrium point of system of equation.

Theorem: The basic reproduction number R_0 for the system of equation (1.1) is given by $R_0=\gamma_2\gamma_0/d\beta_0$.

Proof: Rewriting the given system of equation (1.1)

$$\begin{aligned} ^cD^\delta y &= y(\gamma_2 x - d - \alpha_1 z_2), \\ ^cD^\delta z_1 &= \alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1, \\ ^cD^\delta z_2 &= \alpha_3 z_1 - d_2 z_2, \\ ^cD^\delta x &= x(\gamma_0 - \beta_0 x - \gamma_1 y), \end{aligned}$$

system 3.4 can be written as

$^cD^\delta X(t) = f(X) - \nu(X)$

Where, $f(X) = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \gamma_2 y x \\ 0 \\ 0 \\ 0 \end{pmatrix}, \nu(X) = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix} = \begin{pmatrix} dy + \alpha_1 y z_2 \\ -\alpha_2 y z_2 + \alpha_3 z_1 + d_2 z_1 \\ -\alpha_3 z_1 + d_2 z_2 \\ -\gamma_0 x + \beta_0 x^2 + \gamma_1 x y \end{pmatrix}$

Now the Matrices F(X) and V(X) can be defined as

$$F(X) = \begin{pmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial x} \\ \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial z_2} & \frac{\partial f_3}{\partial x} \\ \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z_1} & \frac{\partial f_4}{\partial z_2} & \frac{\partial f_4}{\partial x} \end{pmatrix}, V(X) = \begin{pmatrix} \frac{\partial \nu_1}{\partial y} & \frac{\partial \nu_1}{\partial z_1} & \frac{\partial \nu_1}{\partial z_2} & \frac{\partial \nu_1}{\partial x} \\ \frac{\partial \nu_2}{\partial y} & \frac{\partial \nu_2}{\partial z_1} & \frac{\partial \nu_2}{\partial z_2} & \frac{\partial \nu_2}{\partial x} \\ \frac{\partial \nu_3}{\partial y} & \frac{\partial \nu_3}{\partial z_1} & \frac{\partial \nu_3}{\partial \nu_2} & \frac{\partial \nu_3}{\partial x} \\ \frac{\partial \nu_4}{\partial y} & \frac{\partial \nu_4}{\partial z_1} & \frac{\partial \nu_4}{\partial z_2} & \frac{\partial \nu_4}{\partial x} \end{pmatrix}$$

Then one get,

$$F(X) = \begin{pmatrix} \gamma_2 x & 0 & 0 & \gamma_2 y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V(X) = \begin{pmatrix} d + \alpha_1 z_2 & 0 & \alpha_1 y & 0 \\ -\alpha_2 z_2 & \alpha_3 + d_1 & -\alpha_2 y & 0 \\ 0 & -\alpha_3 & d_2 & 0 \\ \gamma_1 x & 0 & 0 & -\gamma_0 + 2\beta_0 x + \gamma_1 y \end{pmatrix}$$

To obtain the eigenvalues of pest extinction equilibrium point $E_1(\frac{\gamma_0}{\beta_0}, 0, 0, 0)$, the equation $|FV^{-1} - \lambda I| = 0$ has to be solved, λ is the eigenvalue I is the identity matrix, FV^{-1} is the next generation matrix for model (3.4). $\lambda_1, \lambda_2, \lambda_3$ and λ_4 can be computed as $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ and $\lambda_4 = \frac{\gamma_2\gamma_0}{d\beta_0}$. The spectral radius of matrix $F.V^{-1}$ is $\rho(F.V^{-1}) = max(\lambda_i)$, i=1,2,3,4.

By theorem 3 in [16] the basic reproduction number of model (1.1) is $\mathbb{R}_0 = \frac{\gamma_2\gamma_0}{d\beta_0}$. □

Steady state points

- The steady state point E_0 (0, 0, 0, 0) always present.
- The steady state point E_1 (γ_0/β_0, 0, 0, 0) is present.
- Biological enemy free equilibrium point E_2 (d/γ_2 , $1/\gamma_1$ ($\gamma_0-\beta_0 d/\gamma_2$), 0, 0) is exist only when the basic reproduction number $R_0>1$.

4. The coexisting equilibrium point $E_3(x', y', z_1', z_2')$ exist only when C_0 holds. Whe
C_0: $\gamma_2\gamma_0\alpha_2\alpha_3 > \gamma_2\gamma_1 d_2\alpha_3 + \gamma_2\gamma_1 d_2 d_1 + d\beta_0\alpha_2\alpha_3$. and here
$x' = \frac{1}{\alpha_2\alpha_3\beta_0}[\gamma_0\alpha_2\alpha_3 - d_2\gamma_1\alpha_3 - \gamma_1 d_2 d_1]$,
$y' = \frac{(\alpha_3+d_1)d_2}{\alpha_2\alpha_3}$,
$z_1' = \frac{d_2}{\beta_0\alpha_1\alpha_2\alpha_3^2}[\gamma_2\gamma_0\alpha_2\alpha_3 - \gamma_2\gamma_1 d_2\alpha_3 - \gamma_2\gamma_1 d_2 d_1 - d\beta_0\alpha_3\alpha_2]$,
$z_2' = \frac{1}{\beta_0\alpha_1\alpha_2\alpha_3}[\gamma_2\gamma_0\alpha_2\alpha_3 - \gamma_2\gamma_1 d_2\alpha_3 - \gamma_2\gamma_1 d_2 d_1 - d\beta_0\alpha_3\alpha_2]$.

Local stability analysis

The fundamental matrix for the given system of equation (1.1)

$$J(x,y,z_1,z_2) = \begin{pmatrix} \gamma_0 - 2x\beta_0 - \gamma_1 y & -\gamma_1 x & 0 & 0 \\ \gamma_2 y & \gamma_2 x - d - \alpha_1 z_2 & 0 & -\alpha_1 y \\ 0 & \alpha_2 z_2 & -\alpha_3 - d_1 & \alpha_2 y \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The equilibrium point E_0 (0, 0, 0, 0) of system (1.1) is unstable saddle point.

Proof. The Jacobian matrix for equilibrium point E_0 is

$$J(0,0,0,0) = \begin{pmatrix} \gamma_0 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & -\alpha_3 - d_1 & 0 \\ 0 & 0 & 0 & -d_2 \end{pmatrix}$$

The eigen values of the system are $\lambda_1 = \gamma_0, \lambda_2 = -d, \lambda_3 = -\alpha_3 - d_1$, There for e $|arg(\lambda_1)| = 0 < \delta\pi/2$ where, $0 < \delta < 1$ hence by virtue of [21] E_0 is unstable saddle point. □

The equilibrium point E_1 (γ_0/β_0, 0, 0, 0) is locally a symptotically stable, when the basic reproduction number R_0<1

$$J(\tfrac{\gamma_0}{\beta_0},0,0,0) = \begin{pmatrix} -\gamma_0 & -\frac{\gamma_1\gamma_0}{\beta_0} & 0 & 0 \\ 0 & \frac{\gamma_2\gamma_0}{\beta_0} - d & 0 & 0 \\ 0 & 0 & -\alpha_3 - d_1 & 0 \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The eigenvalues for the equilibrium point E_1 are $\lambda_1 = -\gamma_0, \lambda_2 = -d + \frac{\gamma_0\gamma_2}{\beta_0}, \lambda_3 = -\alpha_3 - d_1, \lambda_4 = -d_2$. The argument of the eigenvalues are $|arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_3)| = \pi > \frac{\delta\pi}{2}$ only when $\mathbb{R}_0 < 1$, and $|arg(\lambda_4)| = \pi > \frac{pi}{2}$. Hence by the virtue of [21] the equilibrium point E_1 is locally asymptotically stable. □

Proof. The Jacobian matrix for E_1 is

$$J(\tfrac{\gamma_0}{\beta_0},0,0,0) = \begin{pmatrix} -\gamma_0 & -\frac{\gamma_1\gamma_0}{\beta_0} & 0 & 0 \\ 0 & \frac{\gamma_2\gamma_0}{\beta_0} - d & 0 & 0 \\ 0 & 0 & -\alpha_3 - d_1 & 0 \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The eigenvalues for the equilibrium point E_1 are $\lambda_1 = -\gamma_0, \lambda_2 = -d + \frac{\gamma_0\gamma_2}{\beta_0}, \lambda_3 = -\alpha_3 - d_1, \lambda_4 = -d_2$. The argument of the eigenvalues are $|arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_3)| = \pi > \frac{\delta\pi}{2}$ only when $\mathbb{R}_0 < 1$, and $|arg(\lambda_4)| = \pi > \frac{pi}{2}$. Hence by the virtue of [21] the equilibrium point E_1 is locally asymptotically stable. □

Theorem 3.6. *The biological enemy free equilibrium point $E_2(\frac{d}{\gamma_2}, \frac{1}{\gamma_1}(\gamma_0 - \frac{\beta_0 d}{\gamma_2}), 0, 0)$ if exists then is asymptotically stable when C_1 holds.$C_1: \gamma_2\gamma_0\alpha_2\alpha_3 < \gamma_2\gamma_1 d_1\alpha_3 + \gamma_2\gamma_1 d_2 d_1 + d\beta_0\alpha_2\alpha_3$.*

Proof. The Jacobian matrix for E_1 is

$$J(\tfrac{\gamma_0}{\beta_0},0,0,0) = \begin{pmatrix} -\gamma_0 & -\frac{\gamma_1\gamma_0}{\beta_0} & 0 & 0 \\ 0 & \frac{\gamma_2\gamma_0}{\beta_0} - d & 0 & 0 \\ 0 & 0 & -\alpha_3 - d_1 & 0 \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The eigenvalues for the equilibrium point E_1 are $\lambda_1 = -\gamma_0, \lambda_2 = -d + \frac{\gamma_0\gamma_2}{\beta_0}, \lambda_3 = -\alpha_3 - d_1, \lambda_4 = -d_2$. The argument of the eigenvalues are $|arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_3)| = \pi > \frac{\delta\pi}{2}$ only when $\mathbb{R}_0 < 1$, and $|arg(\lambda_4)| = \pi > \frac{pi}{2}$. Hence by the virtue of [21] the equilibrium point E_1 is locally asymptotically stable. □

Theorem 3.6. *The biological enemy free equilibrium point $E_2(\frac{d}{\gamma_2}, \frac{1}{\gamma_1}(\gamma_0 - \frac{\beta_0 d}{\gamma_2}), 0, 0)$ if exists then is asymptotically stable when C_1 holds.$C_1: \gamma_2\gamma_0\alpha_2\alpha_3 < \gamma_2\gamma_1 d_1\alpha_3 + \gamma_2\gamma_1 d_2 d_1 + d\beta_0\alpha_2\alpha_3$.*

Proof. The Jacobian matrix for E_2 is

$$J(\tfrac{d}{\gamma_2}, \tfrac{1}{\gamma_1}(\gamma_0 - \tfrac{\beta_0 d}{\gamma_2}), 0, 0) = \begin{pmatrix} -\frac{d\beta_0}{\gamma_2} & -\frac{d\gamma_1}{\gamma_2} & 0 & 0 \\ \frac{\gamma_2}{\gamma_1}(\gamma_0 - \frac{d\beta_0}{\gamma_2}) & 0 & 0 & \frac{-\alpha_1}{\gamma_1}(\gamma_0 - \frac{d\beta_0}{\gamma_2}) \\ 0 & 0 & -\alpha_3 - d_1 & \frac{\alpha_2}{\gamma_1}(\gamma_0 - \frac{d\beta_0}{\gamma_2}) \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The Eigen equation for the equilibrium point E_2 is $\{\lambda^2 + \frac{d\beta_0}{\gamma_2}\lambda + d(\gamma_0 - \frac{d\beta_0}{\gamma_2}\}\{\lambda^2 + (d_2 + d_1 + \alpha_3)\lambda + d_2(d_1 + \alpha_3) - \frac{\alpha_2\alpha_3\gamma_0}{\gamma_1} + \frac{d\beta_0\alpha_2\alpha_3}{\gamma_1\gamma_2}\} = 0$ The eigenvalues for the Jacobian matrix have negative real part when according to Routh-Hurwitz criterion C_1 holds. Then $|arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_3)| = \pi > \frac{\delta\pi}{2}$ and $|arg(\lambda_4)| = \pi > \frac{\delta\pi}{2}$.

Theorem 3.7. *The coexisting equilibrium point $E_3(x', y', z_1', z_2')$ if exist then is asymptotically stable when C_2 Holds.*

Proof. The Jacobian matrix for the equilibrium point C_2 is

$$J(x', y', z_1', z_2') = \begin{pmatrix} \gamma_0 - 2x'\beta_0 - \gamma_1 y' & -\gamma_1 x' & 0 & 0 \\ \gamma_2 y' & \gamma_2 x' - d - \alpha_1 z_2' & 0 & -\alpha_1 y' \\ 0 & \alpha_2 z_2' & -\alpha_3 - d_1 & \alpha_2 y' \\ 0 & 0 & \alpha_3 & -d_2 \end{pmatrix}$$

The Eigen-Equation for this Jacobian Matrix is

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0, \tag{3.5}$$

Where, $M_1 = \beta_0 x' + d_1 + d_2 + \alpha_3, M_2 = \gamma_1\gamma_2 x'y' + (d_1 + d_2 + \alpha_3)\beta_0 x'$, $M_3 = \alpha_1\alpha_2\alpha_3 y'z_2' + \gamma_1\gamma_2 x'y'(d_2 + d_1 + \alpha_3), M_4 = \beta_0\alpha_1\alpha_2\alpha_3 x'y'z_2'$, and $x' = \frac{1}{\alpha_2\alpha_3\beta_0}[\gamma_0\alpha_2\alpha_3 - d_2\gamma_1\alpha_3 - \gamma_1 d_2 d_1], y' = \frac{(\alpha_3 + d_1)d_2}{\alpha_2\alpha_3}$, $z_1' = \frac{d_2}{\beta_0\alpha_1\alpha_2\alpha_3^2}[\gamma_2\gamma_0\alpha_2\alpha_3 - \gamma_2\gamma_1 d_2\alpha_3 - \gamma_2\gamma_1 d_2 d_1 - d\beta_0\alpha_3\alpha_2]$,

The equilibrium point E_3 will be stable according to [21] only when the eigenvalues of (3.5)have negative real parts, according to the Routh-Hurwitz criteria the eigen equation (3.5) has eigen values as negative real parts if the following condition is satisfied.

$$C_2\colon \begin{pmatrix} M_1 M_2 - M_3 > 0, \\ M_1(M_2 M_3 - M_4) - M_3^2 > 0. \end{pmatrix}$$

Then $|arg(\lambda_1)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_2)| = \pi > \frac{\delta\pi}{2}$, $|arg(\lambda_3)| = \pi > \frac{\delta\pi}{2}$ and $|arg(\lambda_4)| = \pi > \frac{\delta\pi}{2}$. □

3.6. *Global Stability*

Theorem 3.8. *The Pest extinction equilibrium point $E_1(\frac{\gamma_0}{\beta_0}, 0, 0, 0)$ is globally asymptotically stable if $\mathbb{R}_0 < 1$.*

Proof. Now consider the Lyapunov function as follows:
$\phi(x, y, z_1, z_2) = \frac{\gamma_2}{\gamma_1}(x - \frac{\gamma_0}{\beta_0} - \frac{\gamma_0}{\beta_0}\ln(\frac{x}{\frac{\gamma_0}{\beta_0}})) + y + \frac{\alpha_1}{\alpha_2}z_1 + \frac{\alpha_1(\alpha_3 + d_1)}{\alpha_2\alpha_3}z_2$

By virtue of lemma (3.1) of [30] the δ order derivative of $\phi(x, y, z_1, z_2)$ is
${}^cD^\delta\phi(x, y, z_1, z_2) \le \frac{\gamma_2}{\gamma_1}(1 - \frac{\gamma_0}{x\beta_0}){}^cD^\delta x +{}^c D^\delta y + \frac{\alpha_1}{\alpha_2}{}^cD^\delta z_1 + \frac{\alpha_1(\alpha_3 + d_1)}{\alpha_2\alpha_3}{}^cD^\delta z_2$
$\le \frac{\gamma_2}{\gamma_1}(1 - \frac{\gamma_0}{x\beta_0})(\gamma_0 - \beta_0 x^2 - \gamma_1 xy) + y(\gamma_2 x - d - \alpha_1 z_2) + \frac{\alpha_1}{\alpha_2}(\alpha_2 y z_2 - \alpha_3 z_1 - d_1 z_1) + \frac{\alpha_1(\alpha_3 + d_1)}{\alpha_2\alpha_3}(\alpha_3 z_1 - d_2 z_2)$
$\le -\frac{\gamma_2}{\gamma_1\beta_0}(x - \frac{\gamma_0}{\beta_0})^2 + d(\mathbb{R}_0 - 1)y - \frac{\alpha_1(\alpha_3 + d_1)}{\alpha_2\alpha_3}d_2 z_2$
Hence ${}^cD^\delta(\phi) \le 0$ when $\mathbb{R}_0 < 1$ from Lemma (4.6) in [31] the pest extinction equilibrium point is globally asymptotically stable. □

Numerical analysis

In this section we have developed some graphs of the system (1.1) by using Matlab code FDE12. We will use the values of the parameter as mentioned in the Table 1 and initial value of the population is taken as [x(0), y(0), z_1 (0), z_2(0)]=[1, 1, 1, 1].

TABLE 1
A table for the various values of parameters used in system (1.1)

Para.↓	Coll	Col2	Col3	Col4
γ_0	0.9	0.9	0.9	0.3
γ_1	0.1	0.1	0.1	0.1
γ_2	0.02	0.2	0.2	0.2
α_1	0.3	0.3	0.3	0.3
α_2	0.1	0.1	0.25	0.1
α_3	0.25	0.25	0.25	0.25
β_0	0.1	0.1	0.1	0.1
d	0.2	0.2	0.2	0.2
d_1	0.1	0.1	0.2	0.1

d_2	0.2	0.2	0.1	0.2

Pest extinction point

Taking the values of parameters as defined in column-1 of Table 1, one can observe that the progress rate of pest population (γ_2=0.02) is very less than the other cases and the basic reproduction number R_0=0.9<1 then we get the pest extinction equilibrium point which is shown in following graphs (Figures 1 and 2).

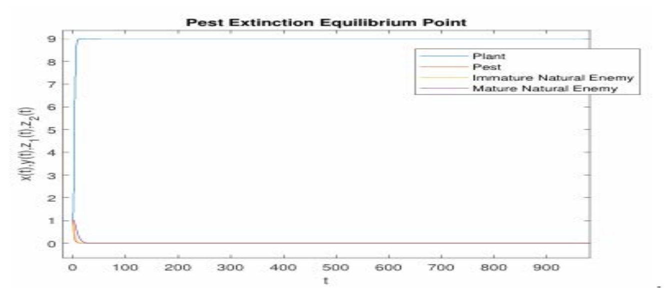

Figure 1) *Time series graph of pest extinction equilibrium point E_0 (9, 0, 0, 0) for δ=1, this graph shows that when there is no pest and biological enemy then plant population is higher than the other cases as discussed in this paper*

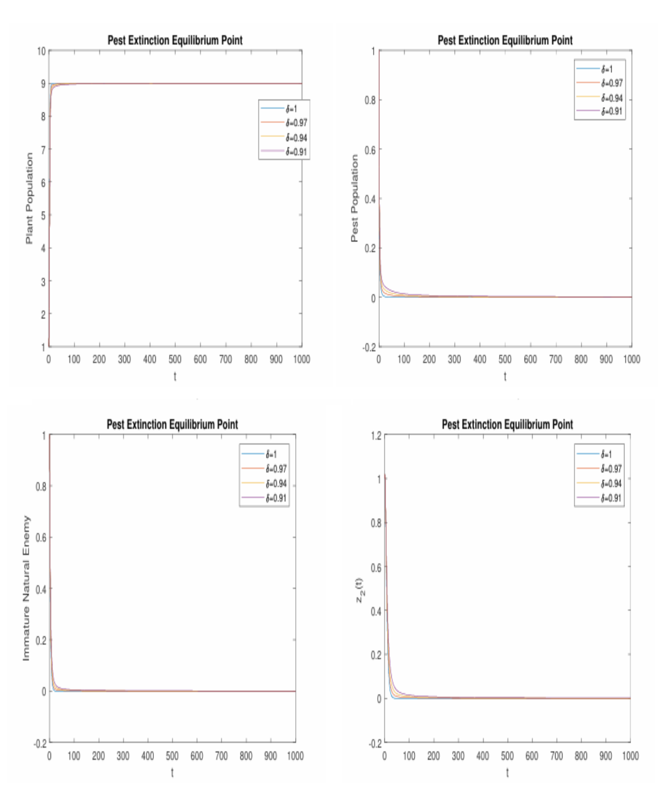

Figure 2) *Set of point of state space for the system 1.1, for the different values of fractional parameter δ=1, 0.97, 0.94, 0.91 as defined in Table 1*

Coexisting point

One can observe from column-2 of Table 1 that when we take $\gamma 2$=0.2 then we get the R0=9>1 which shows the existence of coexisting equilibrium point (Figures 3-5).

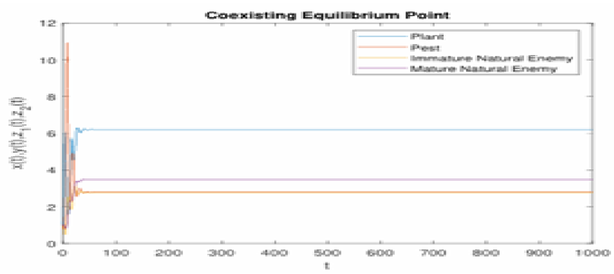

Figure 3) *Time series graph of coexisting equilibrium point E_3 (6.2, 2.8, 2.7733, 3.4667) for δ=1, and one can observe from the graph that the plant population in presence of pest and biological enemy is higher than the situation when there are no biological enemies and is lesser to the situation when there is no pest and biological enemy both*

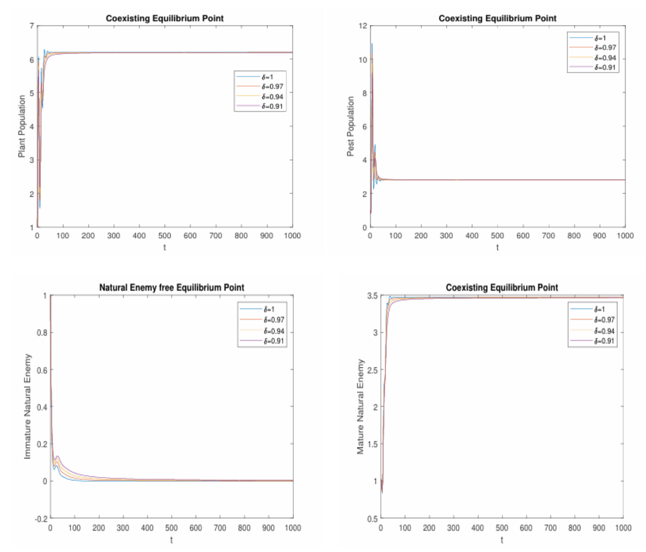

Figure 4) *Set of point of state space for the system 1.1, for coexisting equilibrium for the different values of fractional parameter δ=1, 0.97, 0.94, 0.91 as defined in column-2 of Table 1*

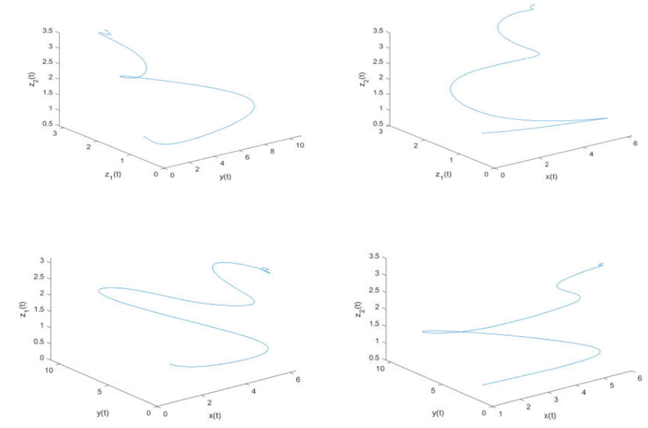

Figure 5) *Portrait diagram of the model for δ=0.97 with reference to the column-2 of Table 1*

Unstable coexisting point

One can observe from column-3 of Table 1 that if one takes the mortality rate (d_1=0.2) of immature biological enemy higher than the mortality rate (d_2=0.1) of mature biological enemy and the progress rate of immature biological enemy (α_2=0.25) then the basic reproduction number is R_0=9>1 the coexisting equilibrium point becomes unstable (Figures 6-8).

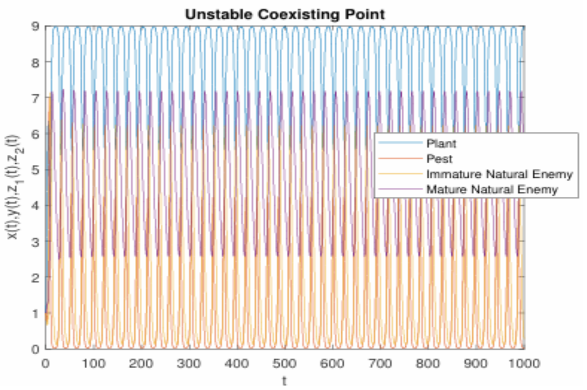

Figure 6) *Time series graph of unstable coexisting equilibrium point E_3 (8.9724, 0.0228, 0.2802, 5.0282) for δ=1 and other parameters as defined in column-3 of Table 1*

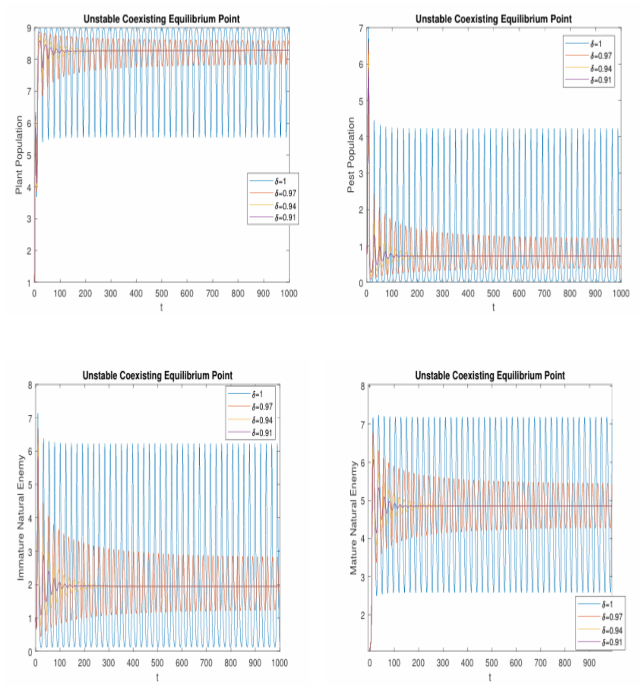

Figure 7) *Set of point of state space for the system 1.1, for unstable coexisting equilibrium for the different values of fractional parameter δ=1, 0.97, 0.94, 0.91 and other parameters as defined in column-3 of Table 1*

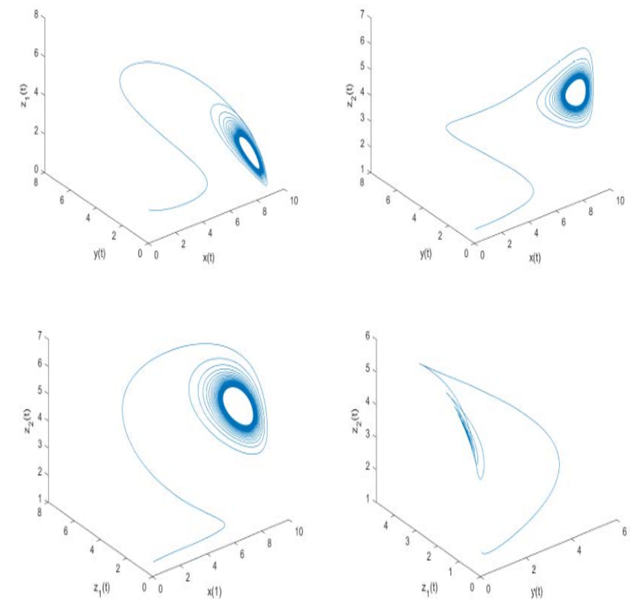

Figure 8) *Portrait diagram of the model (1.1) for δ=0.97 with reference to the column-3 of Table 1*

Biological enemy free point

One can observe from the column-4 of Table 1 that when the growth rate plants (γ_0=0.3) is less than the other cases then R_0=3>1 this shows that there is an existence of pest population but one gets the biological enemy free equilibrium state as shown in the following graphs (Figure 9).

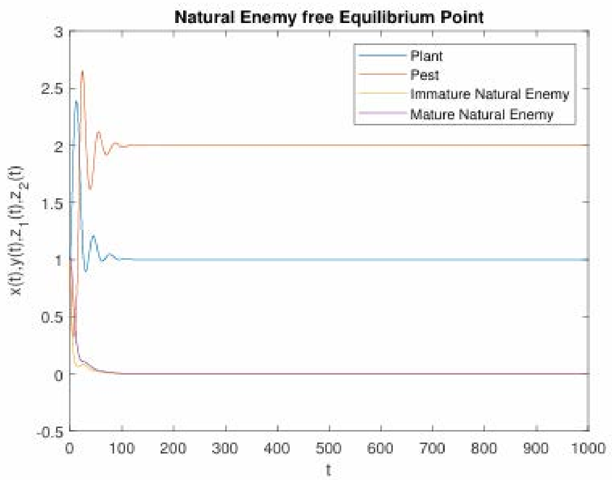

Figure 9) *Time series graph of biological enemy free equilibrium point E_2 (1, 2, 0, 0) for δ=1*

One can see from the graph that the plant population in presence of pest and without biological enemy is lesser than all the cases we discussed in this paper. Hence if there is exist pest in plant than there should by biological enemy for the protection and growth of plants (Figure 10).

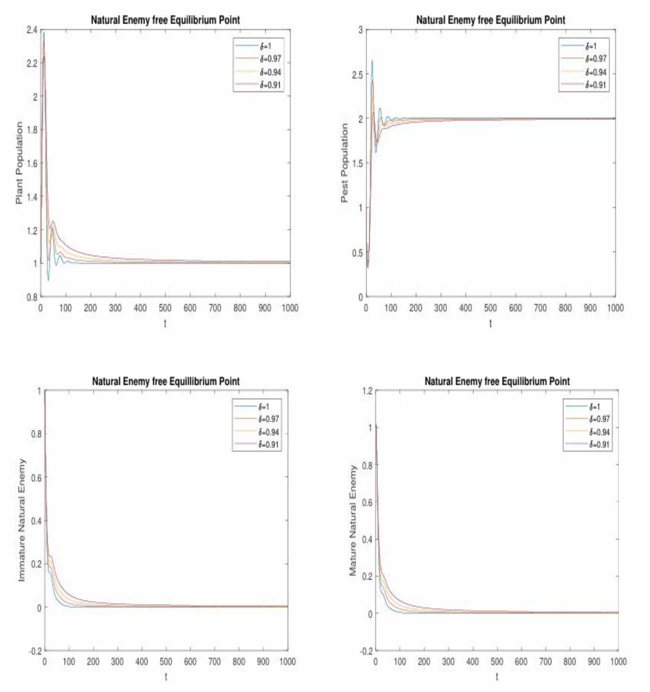

Figure 10) *Set of point of state space for the system 1.1, for biological enemy free equilibrium for the different values of fractional parameter* δ*=1, 0.97, 0.94, 0.91 as defined in column-4 of Table 1*

CONCLUSION

In this paper, we presented a result on the existence and uniqueness, of the solution as well as (3.2) on the non-negativity and uniform boundedness for a class of systems under the control of (1.1). The stability of the equilibrium points has been discussed. According to the discussion in (3.4), the equilibrium point E_0 is unstable saddle point, the equilibrium point E_1 is locally and globally asymptotically stable when the condition $R_0<1$ holds as discussed in 3.5 and 3.8, the equilibrium point E_2 is asymptotically stable only when the condition C_1 holds as discussed in 3.6 and the equilibrium point E_3 is also locally and globally asymptotically stable If C_2 holds as discussed in 3.7. In the end, equilibrium points are numerically analysed as explained in (4). From the numerical simulation, it can be seen that fractional order changes the convergence speed of the solution of fractional differential system and it is also seen when fractional order δ increases ($0<\delta<1$) the convergence speed of solution is also increased which shows the memory term of fractional order.

DATA AVAILABILITY

The labelled dataset used to support the findings of this study is available from the corresponding author on request.

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